On the Microtheoretic Foundations of Cagan's Demand for Money Function.*

Rajat Deb Kaushal Kishore
Tae Kun Seo
Department of Economics
Southern Methodist University
Dallas, TX 75275, U.S.A.

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1. Introduction

An extensive literature, both theoretical[1] and empirical,[2] has arisen around the special semi-logarithmic demand for money function introduced by (Cagan 1956). Cagan's motivation behind the demand for money function was mainly in terms of transactions costs and its relationship to the consumer's ability to affect the real value of cash balances. Cagan argued that the real cost of holding cash balances fluctuates widely enough to account for the dramatic changes in the holding of cash balances observed during hyperinflation. He hypothesized that during periods of hyperinflation the demand for money is almost entirely explained by the variation in the expected rate of change in prices and that changes in expected inflation have the same effect on real balances in percentage terms regardless of the absolute amount of initial cash balances. In other words, during hyperinflations, the demand for money takes the special form: $m = ke^{-\lambda\pi^e}$, where m is the real demand for money, π^e is the expected rate of inflation and k, λ are positive constants.

The theoretical papers using Cagan's functional form have been written largely in the monetarist tradition, analyzing hyperinflation and the associated problem of "inflation tax."(See, for instance, Calvo and Lederman 1992, Sargent and Wallace 1973, Friedman 1971.) The use of Cagan's demand for money function has, however, been "ad-hoc" and no attempt has been made to rationalize the function in terms of "utility maximizing" behavior. This paper examines the possibility of providing such a rationalization, without introducing money directly into the consumer's utility function. We assume that individuals are rational and that money is both a medium of exchange and a store of value, and that the demand for money is a result of intertemporal consumption smoothing. In this framework we try to solve the so called "integrability problem" by asking the question as to whether Cagan's special semi-logarithmic form of the demand for money can be generated from some underlying process of utility maximization.

We provide the answer to this question in the context of two different models. The first is a simple two-period utility maximizing model of the type used extensively in "overlapping generations" literature in macroeconomics (Samuel-

[1] See for instance, Calvo and Leiderman 1992, Bruno and Fisher 1990, Goldman 1974, Sargent and Wallace 1973, Friedman 1971.

[2] See for instance, Metin and Muslu 1999, Easterly, Mauro and Schmidt-Hebbel 1995, Michael, Nobay and Peel 1994, Engsted 1993, Taylor 1991, Anderson, Bomberger and Makinen 1988, Christiano 1987, Salemi and Sargent 1979, Aghevli and Khan 1977, Babcock and Makinen 1975, Pickersgill 1968, Cagan 1956.

son 1958, Diamond 1965). The second is a transactions cost/inventory theoretic model of the "Baumol-Tobin" type (Baumol 1952, Tobin 1956). For the first type of model we discuss and analyze the type of utility function that gives rise to Cagan's demand for money function. We show that while the function has the "usual" properties assumed in utility theory, no *time separable* utility function *of the type usually used* in overlapping generations models can generate Cagan's form for the demand for money. For the second type of model, we show that in a "Baumol-Tobin" type of inventory theoretic framework, the demand for money takes Cagan's form if and only if the transactions cost function in the model takes a specific form. Our results are shown to be valid both in static and fully dynamic versions of this model.

2. Demand for Money: Cagan's Functional Form

Let m be the real quantity of money, M the nominal quantity of money and P the price level. The usual demand for money function used in macroeconomics posits a positive relation between the real demand for money, $m_t \equiv \frac{M_t}{P_t}$, and real income, y_t, and a negative relationship between the real demand for money and the nominal interest rate, i_t. A special semi-logarithmic form of this relationship may be written as, $\ln m_t = \widetilde{k} + \gamma \ln y_t - \lambda i_t$ or equivalently as:

$$\frac{M_t}{P_t} = ke^{-\lambda i_t} y_t^\gamma \tag{2.1}$$

where k, γ and λ are positive constants.

Using Fisher's equation, $i_t = r_t + \pi_t^e$, relating the nominal interest rate to the real interest rate, r_t, and the expected inflation rate, π_t^e, one can think of two types of regimes. Firstly, we can have macroeconomic regimes with stable prices where the real interest rate is constant and is primarily determined by the marginal product of capital. Since prices are stable, nominal interest rate is constant too. In such a regime, $\frac{M_t}{P_t} = \widehat{k} y_t^\gamma$ where $\widehat{k} \equiv ke^{-\lambda i_t} > 0$ is a positive constant. If $\gamma = 1$, one gets the classical "quantity theory" of money. A second type of scenario is that of an inflationary environment such as those studied by Cagan (1956) in which hyperinflation prevailed and real income stagnated. In this case, maintaining the assumption that the real rate of interest does not change and is approximately zero, since real income does not change as well, the demand for money takes Cagan's special form and is given by:

$$\frac{M_t}{P_t} = k_0 e^{-\lambda \pi_t^e} \tag{2.2}$$

where $k_0 = k y_t^{\gamma}$ .While this particular functional form for the demand for money has been extremely useful in empirical analyses of money demand during inflationary periods, the following open question remains: Can this form arise from the utility maximizing behavior of a representative agent? We will address this question in the context of two types of standard models used in macroeconomics, the Samuelson-Diamond two-period overlapping generations model and the Baumol-Tobin transactions cost/inventory theoretic model.

3. Model A: The Overlapping Generations Model

Consider a simple model with one good where a representative agent lives for two periods. The agent's utility function is given by $u(c_1, c_2)$ where c_1 and c_2 represent the agent's consumption in periods 1 and 2, respectively. Money does not enter the utility function and thus has no intrinsic value. The agent receives (real) income, y in the first period. No income is earned in the second period. Consumption in the second period is paid from savings held in the form of money, and the demand for money is thus "derived demand" motivated by consumption smoothing. Let $p_1,\ p_2 > 0$ be the price of the good in the first and the second periods, respectively, and M the nominal quantity of money. Then, the agent's utility maximization problem can be written as:

$$\max_{c_1, c_2} u(c_1, c_2) \tag{3.1}$$

such that

$$M + p_1 c_1 = p_1 y \equiv Y \tag{3.2}$$

$$p_2 c_2 = M \tag{3.3}$$

$$c_1, c_2 \geq 0$$

The model that is usually used in macroeconomics is in fact a special case of the above model. In the "standard" overlapping generations model it is generally assumed that the utility function u is additively time separable and that it can be written as sum of utility from consumption in period one and the discounted value of the utility from consumption in period two. Letting the discount factor be $(1+\theta)^{-1}$ with $\theta > 0$, the agent's maximization problem can, in this case, be rewritten as:

$$\max_{c_1, c_2} \left[u\left(c_1\right) + \left(1+\theta\right)^{-1} u\left(c_2\right) \right] \tag{3.4}$$

such that

$$\begin{aligned} p_2 c_2 + p_1 c_1 &= p_1 y \equiv Y \\ p_2 c_2 &= M \end{aligned}$$

$$c_1, c_2 \geq 0$$

Under the standard assumption that the agent's expectation is "rational" (Lucas 1972) and that the expected price in period 2 is the same as that predicted by the agent, the inflation and expected inflation rates, π and π^e are identical and $\pi^e = \pi \equiv \frac{p_2}{p_1} - 1$. Using the unit income elastic version of Cagan's demand for money function (2.2), (i.e., assuming $\gamma = 1$), we have:

$$M = k_0 e^{-\lambda \pi^e} Y, \;\; k_0 > 0, \tag{3.5}$$

Solving for $c_1$and c_2 from (3.2) and (3.3) we get,

$$c_1 = \frac{Y}{p_1} \left[1 - k_0 e^{-\lambda \pi^e} \right] \tag{3.6}$$

$$c_2 = \frac{Y}{p_2} \left(k_0 e^{-\lambda \pi^e} \right) \tag{3.7}$$

We will resolve two issues of rationalizability. First, we will demonstrate that there exists a "well behaved" utility function that generate c_1 and c_2 as described in equations (3.6) and (3.7) as interior solutions for the utility maximization problem described by (3.1), (3.2) and (3.3). Second, we will prove that the utility function that rationalizes Cagan's demand for money function *has no (differentiable) monotonic transformation* that is additively separable. This will establish that Cagan's form cannot be generated as the solution to the utility maximization problem of the "standard" model described by equations (3.4), (3.2) and (3.3).

To describe a utility function which can rationalize Cagan's form we will introduce two additional functions: g and its inverse h.

Define a function $g : ((\max\{0, 1 + \lambda^{-1} \ln k_0\}, \infty) \to (0, \infty)$ as:

$$g(\xi) = \xi \left[k_0^{-1} e^{\lambda(\xi-1)} - 1\right] \tag{3.8}$$

It is easy to check[3] that both g and its derivative g' are positive and hence $h \equiv g^{-1}$, the inverse of g, is a well defined continuous function from $(0, \infty)$ to $(0, \infty)$.

Now, let consumption be given by (3.6) and (3.7). Then, we have

$$\frac{c_1}{c_2} = g(\frac{p_2}{p_1}) = \frac{p_2}{p_1}[k_0^{-1} e^{\lambda(\frac{p_2}{p_1}-1)} - 1]$$

Normalizing the price of the first period consumption to be 1, we can write, $y = \frac{Y}{p_1}$ and $p = \frac{p_2}{p_1}$. Using the budget constraint, we get:

$$c_2 = k_0 \frac{y}{p} e^{-\lambda(p-1)}$$

Invoking the integrability condition, we have:

$$\frac{d\mu}{dp} = k_0 \frac{\mu}{p} e^{-\lambda(p-1)} \tag{3.9}$$

where for the indirect utility function ν, μ is the expenditure function, $\mu = \mu(p; \nu(q, Y))$. Thus, μ gives us the minimum expenditure needed when the price vector is p to obtain the maximum utility when the income is y and the price vector is q. Therefore, from (3.9):

$$\ln \mu = \int_q^p \frac{k_0}{t} e^{-\lambda(t-1)} dt + \ A$$

Note that $\mu = y$ if $p = q$ and that the constant of integration $A = \ln y$. Hence,

$$\mu = y e^{\int_q^p \frac{k_0}{t} e^{-\lambda(t-1)} dt}$$

where $p = \frac{p_2}{p_1}$ and $y = \frac{Y}{p_1}$. Now, fixing $p = \frac{p_2}{p_1} = \beta > 1$ and normalizing income to be 1, $\overline{q}_i = \frac{q_i}{Y}$ for $i = 1, 2$ gives us

$$\mu = \frac{1}{\overline{q}_1} e^{\int_{\frac{\overline{q}_2}{\overline{q}_1}}^{\beta} \frac{k_0}{t} e^{-\lambda(t-1)} dt}$$

Noting that $\sum \overline{q}_i c_i = 1$, we have $\frac{1}{\overline{q}_1} = c_1 + \frac{\overline{q}_1}{\overline{q}_2} c_2$.

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[3] Note that $g'(\xi) = k_0^{-1} e^{\lambda(\xi-1)} + \xi \lambda k_0^{-1} e^{\lambda(\xi-1)} - 1$. Since ξ, λ and k_0 are all strictly positive we see that $k_0^{-1} e^{\lambda(\xi-1)} - 1 > 0$ implies that $g > 0$ and $g'(\xi) > 0$.

Thus, $\xi > 1 + \lambda^{-1} \ln k_0$ is sufficient to ensure that $g > 0$ and $g' > 0$. That $\xi > 1 + \lambda^{-1} \ln k_0$ is implied by the domain of g.

Now, substituting $h(\frac{c_1}{c_2})$ for $\frac{\overline{q}_1}{\overline{q}_2}$ we get:

$$\begin{aligned}
\widetilde{u}\,(c_1, c_2) &= [c_1 + c_2 h(\frac{c_1}{c_2})] e^{\int_{h\left(\frac{c_1}{c_2}\right)}^{\beta} k_0\left(\frac{1}{s}\right) e^{-\lambda(s-1)} ds} \qquad (3.10) \\
&= c_2 \left[\frac{c_1}{c_2} + h(\frac{c_1}{c_2})\right] e^{\int_{h\left(\frac{c_1}{c_2}\right)}^{\beta} k_0\left(\frac{1}{s}\right) e^{-\lambda(s-1)} ds} \\
&= c_2 \left[\frac{c_1}{c_2} + h(\frac{c_1}{c_2})\right] e^{\sigma(\frac{c_1}{c_2})} \qquad (3.11)
\end{aligned}$$

where $\sigma(\frac{c_1}{c_2}) \equiv \int_{h\left(\frac{c_1}{c_2}\right)}^{\beta} k_0 \left(\frac{1}{s}\right) e^{-\lambda(s-1)} ds$ for some constant $\beta > \max\{0, 1+\lambda^{-1} \ln k_0\}$.

Proposition 1. *The utility function $\widetilde{u}$ in (3.10) is homogeneous of degree 1 in (c_1, c_2) and is strictly quasi-concave with marginal utilities being positive for both c_1 and c_2. If for this utility function an interior solution to the utility maximization problem (3.1) exists, then the demand for c_1 and c_2 are given by (3.6) and (3.7) and hence the demand for money is given by Cagan's demand for money function (3.5).*

Proof. Let

$$\phi(\frac{c_1}{c_2}) \equiv \left[\frac{c_1}{c_2} + h(\frac{c_1}{c_2})\right] e^{\int_{h\left(\frac{c_1}{c_2}\right)}^{\beta} k_0\left(\frac{1}{s}\right) e^{-\lambda(s-1)} ds}$$

and note that our utility function (3.10) can be written as:

$$\widetilde{u}\,(c_1, c_2) = c_2 \phi(\frac{c_1}{c_2}). \qquad (3.12)$$

From (3.12) it is obvious that $\widetilde{u}$ is homogeneous of degree 1 in (c_1, c_2). Denoting $\frac{c_1}{c_2}$ by x and using (3.10), the marginal utilities of c_1 and c_2 are given by:

$$\widetilde{u}_1 = \phi'(x) \text{ and } \widetilde{u}_2 = \phi(x) - x\phi'(x) \qquad (3.13)$$

By our definitions of g and h, $\xi = h(x)$ if and only if $x = g(\xi) = \xi\left[k_0^{-1}e^{\lambda(\xi-1)} - 1\right]$. Hence, $x = h(x)\left[k_0^{-1}e^{\lambda(h(x)-1)} - 1\right]$. This gives us:

$$x + h(x) = k_0^{-1}h(x)e^{\lambda(h(x)-1)} \tag{3.14}$$

Using (3.14), observe[4] that $\phi(x) - \phi'(x)x = \phi'(x)h(x)$. Hence, (3.13) gives us:

$$\widetilde{u}_2 = \widetilde{u}_1 h(\frac{c_1}{c_2}) \tag{3.15}$$

Assume to the contrary that $\widetilde{u}_1 \leq 0$. Now, by (3.13), if $\widetilde{u}_1 \leq 0$, $\widetilde{u}_2 > 0$. Since, $h > 0$, this contradicts (3.15). Thus, $\widetilde{u}_1 > 0$ and (using (3.15)) $\widetilde{u}_2 > 0$.

Furthermore, by (3.15) along any indifference curve, $\frac{dc_1}{dc_2} = -h(\frac{c_1}{c_2})$. Hence, we get

$$\frac{d^2c_1}{dc_2^2} = -h'(\frac{c_1}{c_2})\left[\frac{c_2\frac{dc_1}{dc_2} - c_1}{c_2^2}\right] = -h'(\frac{c_1}{c_2})\left[\frac{-h(\frac{c_1}{c_2})c_2 - c_1}{c_2^2}\right] > 0$$

This establishes that the utility function is strictly quasi-concave.

Finally, using (3.15) and writing down the first order condition for an interior solution, we have: $h(\frac{c_1}{c_2}) = \frac{p_2}{p_1}$. In other words, $\frac{c_1}{c_2} = g(\frac{p_2}{p_1}) \equiv \frac{p_2}{p_1}\left[k_0^{-1}e^{\lambda\left(\frac{p_2}{p_1}-1\right)} - 1\right]\frac{p_2}{p_1}$. It is easy to verify, that (3.6) and (3.7) satisfy this condition. Hence, using strict quasi-concavity, the unique interior solution to the utility maximizing problem will yield a demand for money function having Cagan's form. ■

To understand when an interior solution to our utility maximizing problem will exist note that as $\frac{c_1}{c_2} \longrightarrow 0$, $\frac{\widetilde{u}_2}{\widetilde{u}_1} \longrightarrow \max\left[0, 1 + \lambda^{-1}\ln k_0\right]$ and as $\frac{c_1}{c_2} \longrightarrow \infty$, $\frac{\widetilde{u}_2}{\widetilde{u}_1} \longrightarrow \infty$. This implies that two types of indifference curves are possible. Case (a): $1 + \lambda^{-1}\ln k_0 \leq 0$: in this case, $\frac{\widetilde{u}_2}{\widetilde{u}_1} \longrightarrow 0$ as $\frac{c_1}{c_2} \longrightarrow 0$, and the indifference curves do not intersect either axis. Case (b): $1 + \lambda^{-1}\ln k_0 > 0$: in this case,

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[4]

$$\begin{aligned}\phi'(x) &= [1 + h'(x)]\frac{\phi(x)}{x + h(x)} - \phi(x)\frac{k_0 e^{-\lambda(h(x)-1)}h'(x)}{h(x)} \\ &= \phi(x)\left[\frac{1 + h'(x)}{x + h(x)} - \frac{h'(x)}{x + h(x)}\right] \text{ by (3.14)} \\ &= \frac{\phi(x)}{x + h(x)}.\end{aligned}$$

$\frac{\widetilde{u}_2}{\widetilde{u}_1} \longrightarrow 1 + \lambda^{-1} \ln k_0$ as $\frac{c_1}{c_2} \longrightarrow 0$, thus, while indifference curves do not cut the c_1 axis, they do intersect the c_2 axis. In particular, this implies that an interior solution exists in this case if and only if $\frac{p_2}{p_1} > 1 + \lambda^{-1} \ln k_0$. The two cases are illustrated in the figure below:

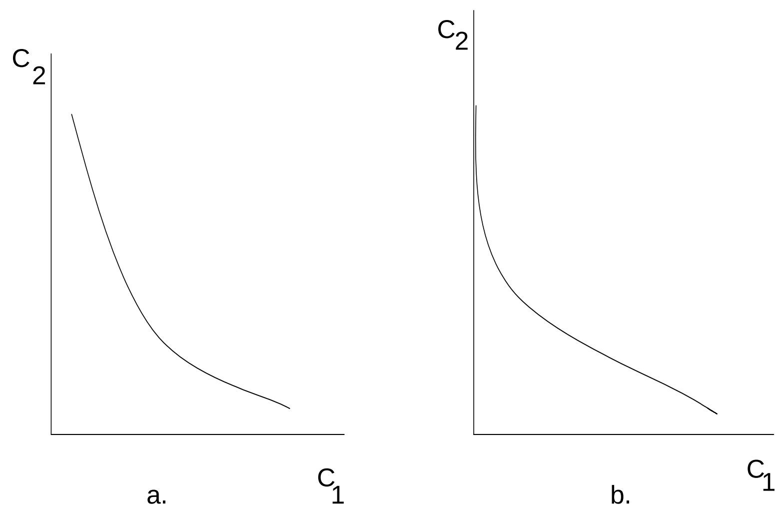

Figure 3.1:

In Case (a) an interior solution will exist for all positive values of p_1 and p_2. Case (b) on the other hand implies that an interior solution exists (and hence, Cagan's form of the money demand function is appropriate) if and only if $\frac{p_2}{p_1} - 1 > \lambda^{-1} \ln k_0$ (i.e., *if and only if the rate of inflation is high enough*). Which of these cases prevails depends on empirical values of the parameters λ and k_0. It is interesting to note that estimates in empirical studies suggest that $1 + \lambda^{-1} \ln k_0 > 0$ and that Case (b) is the more plausible of the two cases. (See for instance, Metin and Muslu 1999, Easterly, Mauro and Schmidt-Hebbel 1995, Michael, Nobay and Peel 1994, Engsted 1993, Taylor 1991, Anderson, Bomberger and Makinen 1988, Christiano 1987, Salemi and Sargent 1979, Aghevli and Khan 1977, Babcock and Makinen 1975, Pickersgill 1968, Cagan 1956.)

We have provided an example of an utility function, $\widetilde{u}$, that rationalizes Cagan's form of the demand for money function. Clearly, a necessary and sufficient condition for any utility function to generate this demand function is that it be a strictly monotonic transformation of $\widetilde{u}$. Now, using this property, we turn to the question of rationalizing Cagan's demand function in the "standard" version of the overlapping generations model with a time separable utility function.

Proposition 2. *There does not exist a function $u : \mathcal{R}_+ \to \mathcal{R}$ and a differentiable strictly monotonic transformation v of $\tilde{u}$ defined by (3.10) such that (i) $v(c_1, c_2) = u(c_1) + (1+\theta)^{-1} u(c_2)$ and (ii) v gives rise to Cagan's form of the demand for money function.*

Proof. Assume to the contrary that v is such a monotonic transformation. Then, $\ln \frac{v_1}{v_2} = \ln \frac{u_1}{(1+\theta)^{-1} u_2} = \ln u_1(c_1) - \ln (1+\theta)^{-1} u_2(c_2)$. This implies that $\frac{\partial}{\partial c_1}[\frac{\partial}{\partial c_2} \ln \frac{v_1}{v_2}] \equiv 0$.[5] Since v is a monotone transformation, we would have $\frac{\partial}{\partial c_1}[\frac{\partial}{\partial c_2} \ln \frac{\tilde{u}_1}{\tilde{u}_2}] \equiv 0$.

But, from (3.15), $\ln \frac{\tilde{u}_1}{\tilde{u}_2} = \frac{1}{h} = -\ln h$. Hence, we have:

$$\frac{\partial}{\partial c_2} \ln \frac{\tilde{u}_1}{\tilde{u}_2} = \frac{h'}{h} \frac{c_1}{c_2^2}$$

Thus, for $\frac{\partial}{\partial c_1}[\frac{\partial}{\partial c_2} \ln \frac{\tilde{u}_1}{\tilde{u}_2}] \equiv 0$, it must be the case that $\frac{h'}{h} c_1$ is a function of c_2 alone, say, $\frac{h'}{h} c_1 \equiv \psi(c_2) c_2^2$. The left hand side of this equation is homogeneous of degree 1 in (c_1, c_2). This implies that the right hand side is homogenous of degree 1 in c_2. That is $\psi(c_2) c_2^2 \equiv a c_2$ for some *positive* constant a.[6] Thus, we can write:

$$\frac{h'(x)}{h(x)} = \frac{a}{x}$$

Integrating both sides, we have $h(x) \equiv b x^a$, where b is a constant of integration. Thus, $g \equiv h^{-1}(x) = (\frac{1}{b})^a \xi^{\frac{1}{a}}$. But, comparing this with $\xi \left[k_0^{-1} e^{\lambda(\xi-1)} - 1 \right]$ from (3.8), and letting $\xi \to \infty$ we see that $(\frac{1}{b})^a \xi^{\frac{1}{a}} \equiv \xi \left[k_0^{-1} e^{\lambda(\xi-1)} - 1 \right]$ is impossible. ■

4. Model B: The Transactions Cost Model

In this model a representative agent faces two costs: a "transactions cost" and an "opportunity cost of holding money" (see Tobin 1956). If money is held for transactions purposes, then it cannot be invested and the interest, i, that could have been earned is foregone. Thus, the opportunity cost of holding the stock of

[5] See Sono's (1961) classic analysis of separability.

[6] Note that a is not equal to zero since this will imply that h is a constant.

money m is im. The transactions cost is some function of the amount of money being held and the level of transactions. The cost of transactions decreases with the amount of money held, but decreases at a decreasing rate. The real income y is a proxy for the volume of transactions and the transactions cost increases with y. Thus, we will assume, $\alpha = \alpha(m, y)$ is the transactions cost and in the neighborhood of the equilibrium holding of money $\alpha_m < 0$, $\alpha_{mm} > 0$ and $\alpha_y > 0$, where the real quantity of money is $m \equiv \frac{M}{P}$.

To hold an optimal inventory of money the agent minimizes the sum of the "transactions cost" and the "opportunity cost of holding money".

$$\min_m \left[im + \alpha(m, y)\right]$$

The first-order condition for an interior minimum is given by:

$$-\alpha_m(m, y) = i \tag{4.1}$$

Since $\alpha_{mm} > 0$, in the neighborhood of the equilibrium, the first-order conditions is sufficient for a minimum, and using the implicit function theorem, we can solve (locally) for the optimal level of money demand $\hat{m}$ as a function of i and y. Thus, we have:

$$\hat{m} = \hat{m}(i, y) \text{ with } \hat{m}_i < 0\text{; moreover, } \alpha_{my} < 0 \text{ iff } \hat{m}_y > 0.$$

Thus, in equilibrium, the amount of money held will be a decreasing function of real interest rate, and if $\alpha_{my} < 0$, it will be an increasing function of total income of the agent.

Now consider the following special transactions cost function (where γ is a positive constant).

$$\alpha(m, y) \equiv \lambda^{-1}\left[m \ln m - m - \gamma m \ln y - m \ln k\right] + k\lambda^{-1} y^{\gamma} + \tau(y) \tag{4.2}$$

where τ is an arbitrary real valued function of y such that $\tau'(y) \geq 0$.

Replacing $\alpha(m, y)$ with this specific form in (4.1), the optimum amount of money holding can be calculated. This gives us a generalized version of Cagan's demand for money function as follows:[7]

[7] It is worth noting, that if one used $\alpha(m, y) = \frac{\alpha_0 y}{2m}$ with $\alpha_0 > 0$ then (4.1) would give us Baumol's well known "square root" formula with $\hat{m} = (\frac{\alpha_0 y}{2i})^{\frac{1}{2}}$.

$$
\begin{aligned}
-\alpha_m &\equiv -\lambda^{-1}\left[\ln m - \gamma \ln y - \ln k\right] = i. \qquad (4.3)\\
\text{Or, } \hat{m} &= k e^{-\lambda i} y^{\gamma}.
\end{aligned}
$$

It is easy to check that *at the equilibrium* described by (4.3), $\alpha_m < 0$, $\alpha_{mm} > 0$ and $\alpha_y > 0$.

Conversely, assume that we have (4.3). Then, in the neighborhood of the equilibrium, we can define a function $\rho(y)$ such that:

$$\alpha\left(m, y\right) \equiv \lambda^{-1}\left[m \ln m - m - \gamma m \ln y - m \ln k\right] + \rho(y)$$

Since at the equilibrium, $\alpha_y > 0$, we have $\rho'(y) > \gamma m(\lambda y)^{-1}$. Using (4.3), we get $\rho'(y) > \gamma k \lambda^{-1} e^{-\lambda \cdot i} y^{\gamma - 1}$. Since the last inequality holds for all $i > 0$, we must have $\rho'(y) \geq \gamma k \lambda^{-1} y^{\gamma-1}$. Thus, defining $\rho(y) \equiv k\lambda^{-1} y^{\gamma} + \tau(y)$, we see that $\tau'(y) \geq 0$. This gives us the specific transactions cost function (4.2) above. Thus, we have the generalized form $\hat{m} = k e^{-\lambda \cdot i} y^{\gamma}$ of Cagan's money demand function if and only if the transactions cost has the form (4.2) in the neighborhood of the equilibrium m.

Note, however, that from (4.2) we get:

$$\alpha_y = -\gamma m(\lambda y)^{-1} + k\lambda^{-1} y^{\gamma} + \tau'(y)$$

and

$$\alpha_m = \lambda^{-1}\left[\ln m - \gamma \ln y - \ln k\right]$$

and that $\alpha_y > 0$ and $\alpha_m < 0$ only if m is "small enough". Thus, for our specific transactions cost function, money holdings in equilibrium need to be small enough in this sense. This is particularly interesting because during periods when the inflation rate is high, households typically minimize their holdings of real balances by moving into non-money assets.

To see that a similar conclusion can be derived from a fully dynamic infinite horizon representative agent model, consider a model where the agent is maximizing his or her lifetime utility given a budget constraint. The quantity of labor is fixed as is the wage rate, and only consumption enters into the instantaneous utility function, $u_t\left(.\right)$. A higher consumption level in any period is associated with higher utility. Here, once again, money does not (directly) enter the utility function. However, money does affect the consumption level indirectly. Total wealth of an agent at any particular time period is given by the wage earned along with accumulated savings from previous time periods. Savings in any period can be held either as money (which earns no interest) or in the form of a financial asset f earning a real interest, r. Increasing or decreasing money holdings is costly entailing the cost of switching between financial assets and money. We assume that this cost has a quadratic structure and is given by $\frac{\eta}{2}\left[\overset{\bullet}{m}\right]^2$, for some $\eta > 0$. As in our static model, $\alpha\left(m\left(t\right), y\right)$ represents the transactions cost at time t. If θ is the subjective discount rate, then the agent's intertemporal utility maximization problem can then be written as:

$$\max \int_0^\infty u_t\left(c(t)\right) e^{-\theta t} dt$$

subject to

$$rf_t + wl = \dot{f} + c\left(t\right) + \alpha\left(m\left(t\right), y\right) + \frac{\eta}{2}\left[\dot{m}\right]^2 + \frac{\dot{M}}{P}$$

The left hand of the budget constraint represents the agents income: interest earned (rf_t) and wage income (wl). This income can be used either for current consumption $(c\left(t\right))$, paying for transactions costs $(\alpha\left(m\left(t\right), y\right))$, for accumulating (decumulating) interest earning assets $(\dot{f})$ or for increasing (decreasing) real balances $(\frac{\dot{M}}{P})$ and paying the costs associated with changing real balances.

Since $\frac{d}{dt}\left(\frac{M}{P}\right) = \dot{m} = \frac{\dot{M}}{P} - \frac{M}{P}\left(\frac{\dot{P}}{P}\right)$, we can write $\frac{\dot{M}}{P} = \dot{m} + m\pi$. Thus, under perfect foresight, with $\pi = \pi^e$, the budget constraint becomes:

$$rf_t + wl = \dot{f} + c\left(t\right) + \alpha\left(m\left(t\right), y\right) + \frac{\eta}{2}\left[\dot{m}\right]^2 + m(t)\pi^e + \dot{m}$$

Substituting for c_t in the utility function and using this budget constraint, we get:

$$\max \int_0^\infty u_t\left(rf_t + wl - \dot{f} - \alpha\left(m\left(t\right), y\right) - \frac{\eta}{2}\left[\dot{m}\right]^2 - m(t)\pi^e - \dot{m}\right) e^{-\theta t} dt$$

The Euler equations for this problem are given by

$$\frac{\partial u_t}{\partial m} = \frac{d}{dt}\left(\frac{\partial u_t}{\partial \dot{m}}\right) \quad (4.4)$$

$$\frac{\partial u_t}{\partial f} = \frac{d}{dt}\left(\frac{\partial u_t}{\partial \dot{f}}\right) \quad (4.5)$$

Substituting (4.5) into (4.4) we get:

$$\eta\ddot{m} - r\eta\dot{m} - (\alpha_m + r + \pi^e) = 0. \tag{4.6}$$

The characteristic equation of the system is given by:

$$x^2 - rx - \eta^{-1}\alpha_{mm} = 0$$

Under our assumption $\alpha_{mm} > 0$, one of the eigenvalues of the system will be positive and the other negative indicating the existence of a saddle point trajectory converging to the equilibrium given by:

$$(\alpha_m + r + \pi^e) = 0,$$

which is the same as (4.1).

Proposition 3. *The demand for money in equilibrium has a generalized version of Cagan's functional form if and only if in the neighborhood of the equilibrium the transactions cost function $\alpha(m, y)$is given by:*

$$\alpha(m, y) \equiv \lambda^{-1}\left[m \ln m - m - \gamma m \ln y - m \ln k\right] + k\lambda^{-1}y^{\gamma} + \tau(y) \text{ where } \tau'(y) \geq 0.$$

5. Conclusion

We have examined the possibility of rationalizing Cagan's functional form for the demand for money. We have shown that in a two period overlapping generations model this demand for money can be derived from an utility function satisfying the usual properties of differentiability, strict quasi-concavity and positivity of marginal utilities. Empirical estimates of the parameters of the model suggest that the functional form arises if and only if the inflation rate is high enough. However, under the usual assumptions that the utility function is time separable, Cagan's form would not arise in this type of model. An alternative way for rationalizing Cagan's money demand function is by using a dynamic inventory cost theoretic model of the Baumol-Tobin type. We have identified the specific transactions cost structure which would give rise to a generalized form of Cagan's demand for money function as a saddle point trajectory of such a model. Once again we find that the model would be valid only in periods of significant inflation when households prefer other assets and reduce their holdings of money balances.

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